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1997 J. Phys. A: Math. Gen. 30 4677

(http://iopscience.iop.org/0305-4470/30/13/018)

J. Phys. A: Math. Gen. **30** (1997) 4677–4680. Printed in the UK PII: S0305-4470(97)79564-7

h-deformation of *Gr* (2)

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Received 13 November 1996

Abstract. The h-deformation of functions on the Grassmann matrix group $Gr(2)$ is presented via a contraction of $Gr_q(2)$. As an interesting point, we have seen that, in the case of the h-deformation, both R-matrices of $GL_h(2)$ and $Gr_h(2)$ are the same.

In recent years a new class of quantum deformations of Lie groups and algebras, the so-called h-deformation, has been intensively studied by many authors [1–9]. The h-deformation of matrix groups can be obtained using a contraction procedure. We start with a quantum plane and its dual and follow the contraction method of [9].

Consider the q-deformed algebra of functions on the quantum plane [10] generated by x', y' with the commutation rule

$$x'y' = qy'x'. \tag{1}$$

Applying a change of basis in the coordinates of the (1) by use of the following matrix:

$$g = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \qquad f = \frac{h}{q-1} \tag{2}$$

in the limit $q \to 1$ one arrives [9] at

$$xy = yx + hy^2. \tag{3}$$

We denote the quantum h-plane by $R_h(2)$.

Similarly, one obtains the dual quantum h-plane $R_h^*(2)$ as generated by η, ξ with the relations

$$\xi^2 = 0 \qquad \eta^2 = h\eta\xi \qquad \eta\xi + \xi\eta = 0. \tag{4}$$

Let

$$\widehat{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be a Grassmann matrix in $Gr(2)$. All matrix elements of $\widehat{A}$ are Grassmann. We consider linear transformations with the following properties:

$$\widehat{A} : R_h(2) \longrightarrow R_h^*(2) \qquad \widehat{A} : R_h^*(2) \longrightarrow R_h(2). \tag{5}$$

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The action on the points of $R_h(2)$ and $R_h^*(2)$ of $\widehat{A}$ is

$$\begin{pmatrix} \overline{\eta} \\ \overline{\xi} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{6}$$

We assume that the entries of $\widehat{A}$ commute with the coordinates of $R_h(2)$ and anticommute with the coordinates of $R_h^*(2)$. As a consequence of the linear transformations in (5) the vectors

$$\begin{pmatrix} \overline{\eta} \\ \overline{\xi} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$$

should belong to $R_h^*(2)$ and $R_h(2)$, respectively, which impose the following h-anticommutation relations between the matrix elements of $\widehat{A}$:

$$\begin{aligned}
&\alpha\beta + \beta\alpha = h(\alpha\delta + \beta\gamma) \qquad \alpha\gamma + \gamma\alpha = 0 \\
&\beta\gamma + \gamma\beta = h(\delta\gamma - \gamma\alpha) \qquad \beta\delta + \delta\beta = -h(\alpha\delta + \gamma\beta) \\
&\alpha\delta + \delta\alpha = h(\gamma\alpha - \delta\gamma) \qquad \gamma\delta + \delta\gamma = 0 \\
&\alpha^2 = -h\gamma\alpha \qquad \beta^2 = h(\beta\delta - \alpha\beta + h\alpha\delta) \qquad \gamma^2 = 0 \qquad \delta^2 = h\delta\gamma.
\end{aligned} \tag{7}$$

These relations define the h-deformation of functions on the Grassmann matrix group $Gr(2)$, $Gr_h(2)$.

Alternatively, relations (7) can be obtained by the following similarity transformation [9]:

$$\widehat{A}' = g\widehat{A}g^{-1} \tag{8}$$

which in our case gives

$$\begin{aligned}
&\alpha' = \alpha + \frac{h}{q-1}\gamma \qquad \beta' = \beta + \frac{h}{q-1}\left(\delta - \alpha - \frac{h}{q-1}\gamma\right) \\
&\gamma' = \gamma \qquad \delta' = \delta - \frac{h}{q-1}\gamma
\end{aligned} \tag{9}$$

and then taking the $q \to 1$ limit. Here α', β', γ' and δ' are generators of $Gr_q(2)$, which satisfy the following commutation relations [11, 12]:

$$\begin{aligned}
&\alpha'\beta' + q^{-1}\beta'\alpha' = 0 \qquad \alpha'\gamma' + q^{-1}\gamma'\alpha' = 0 \\
&\gamma'\delta' + q^{-1}\delta'\gamma' = 0 \qquad \beta'\delta' + q^{-1}\delta'\beta' = 0 \\
&\alpha'\delta' + \delta'\alpha' = 0 \qquad \alpha'^2 = \beta'^2 = \gamma'^2 = \delta'^2 = 0 \\
&\beta'\gamma' + \gamma'\beta' = (q - q^{-1})\delta'\alpha'.
\end{aligned} \tag{10}$$

Substituting (9) in (10) one obtains the set of relations (7) above.

The algebra (10) is associative under multiplication and the relations in (10) may be also expressed in a tensor product form [11, 12]:

$$R_q\widehat{A}'_1\widehat{A}'_2 = -\widehat{A}'_2\widehat{A}'_1R_q \tag{11}$$

where

$$R_q = \begin{pmatrix} q+q^{-1} & 0 & 0 & 0 \\ 0 & 2 & q^{-1}-q & 0 \\ 0 & q-q^{-1} & 2 & 0 \\ 0 & 0 & 0 & q+q^{-1} \end{pmatrix}. \tag{12}$$

Here, since the matrix elements of $\widehat{A}'$ are all Grassmannian, for the conventional tensor products

$$\widehat{A}'_1 = \widehat{A}' \otimes I \qquad \text{and} \qquad \widehat{A}'_2 = I \otimes \widehat{A}' \tag{13}$$

one can write (no-grading)

$$(\widehat{A}_1)^{ij}{}_{kl} = \widehat{A}^i{}_k \delta^j{}_l \qquad (\widehat{A}_2)^{ij}{}_{kl} = \delta^i{}_k \widehat{A}^j{}_l \tag{14}$$

where δ denotes the Kronecker delta. Note that in the limit $q \longrightarrow 1$ the matrix R_q becomes twice the 4×4 identity matrix. Note also that although the algebra (10) is an associative algebra of the matrix entries of $\widehat{A}$, R_q does not satisfy the quantum Yang–Baxter equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Thus the Yang–Baxter equation is not a necessary condition for associativity (see the paragraph after (19) for other remarks). It is obvious that a change of basis in the $R_h(2)$ leads to the similarity transformation

$$\widehat{A} = g^{-1}\widehat{A}'g \tag{15}$$

for the quantum Grassmann group and the following similarity transformation for the corresponding R-matrix:

$$R_{h,q} = (g \otimes g)^{-1} R_q (g \otimes g). \tag{16}$$

If we define the R-matrix R_h as

$$R_h = \lim_{q \to 1} R_{h,q} \tag{17}$$

after dividing by 2 we obtain

$$R_h = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

Substituting (9) and (16) in (11) we arrive at the $q \to 1$ limit:

$$R_h \widehat{A}_1 \widehat{A}_2 = -\widehat{A}_2 \widehat{A}_1 R_h. \tag{19}$$

Another interesting point is that, although the R-matrices of $GL_q(2)$ and $Gr_q(2)$ are different, in the case of the h-deformation, the R-matrices of $GL_h(2)$ and $Gr_h(2)$ are the same (see [9] for the R-matrix R_h of $GL_h(2)$). In the limit $q \longrightarrow 1$ both the R-matrices of $GL_q(2)$ and $Gr_q(2)$ become the same 4×4 identity matrix. Although the R-matrix R_q of $Gr_q(2)$ does not satisfy the QYBE, the R-matrix R_h of $Gr_h(2)$ does satisfy the QYBE.

Since the entries of $\widehat{A}$ are all Grassmann, a proper inverse cannot exist. However, the left and right inverses of $\widehat{A}$ can be constructed:

$$\widehat{A}_L^{-1} = \begin{pmatrix} \delta + h\gamma & \beta + h\alpha \\ -\gamma & -\alpha \end{pmatrix} \tag{20}$$

$$\widehat{A}_R^{-1} = \begin{pmatrix} -\delta & \beta + h\delta \\ -\gamma & \alpha + h\gamma \end{pmatrix}. \tag{21}$$

It is now easy to show that

$$\widehat{A}_L^{-1}\widehat{A} = \Delta_L \tag{22}$$

$$\widehat{A}\widehat{A}_R^{-1} = \Delta_R \tag{23}$$

where

$$\Delta_L = \beta\gamma + \delta\alpha \qquad \Delta_R = \gamma\beta + \alpha\delta. \tag{24}$$

In this case Δ_L and Δ_R may at least formally be considered as the left and right quantum (dual) determinants, respectively. Note that one can write

$$\Delta_L \widehat{A}_R^{-1} = \widehat{A}_L^{-1} \Delta_R. \tag{25}$$

Final remarks. We know that all the matrix elements of $\widehat{A}$ are Grassmann (odd or fermionic) if $\widehat{A}$ is a Grassmann matrix, i.e. it belongs to $Gr(2)$. Now let $\widehat{A}$ and $\widehat{A}'$ be any two anticommuting matrices (i.e. any element of Grassmann matrices whose elements $\widehat{A}$ anticommutes with any element of $\widehat{A}'$) that satisfy (10). Then, all the matrix elements of a product $A = \widehat{A}\widehat{A}'$ are bosonic (or even) since the elements of the matrix product of two Grassmann matrices are all bosonic. It can also be verified that the matrix elements of A satisfy q-commutation relations of $GL_q(2)$, i.e. for

$$\begin{aligned} A &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ ab &= qba \qquad ac = qca \qquad bc = cb \end{aligned} \tag{26}$$

etc. That is, if

$$\widehat{A}, \widehat{A}' \in Gr_q(2) \quad \Longrightarrow \quad A = \widehat{A}\widehat{A}' \in GL_q(2).$$

In view of these facts, we can say that there may be no coproduct of the form $\Delta(\widehat{A}) = \widehat{A}\dot{\otimes}\widehat{A}$. For this coproduct is invariant under the q-commutation relations (26) of $GL_q(2)$. These facts also prevent the existence of a coproduct of the form $\Delta(\widehat{A}) = \widehat{A}^{t_2}\dot{\otimes}\widehat{A}$ where t_2 is an involution acting on the elements of $\widehat{A}$. Hence a construction of the coproduct along the lines of [13] is also not possible.

Acknowledgments

This work was supported in part by TBTAK, the Turkish Scientific and Technical Research Council.

References

[1] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[2] Ewen H, Ogievetsky O and Wess J 1991 *Lett. Math. Phys.* **22** 297
[3] Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287
[4] Woronowicz S L 1991 *Rep. Math. Phys.* **30** 259
[5] Ohn C H 1992 *Lett. Math. Phys.* **25** 85
[6] Karimipour V 1994 *Lett. Math. Phys.* **30** 87
[7] Kupershmidt B A 1992 *J. Phys. A: Math. Gen.* **25** L1239
[8] Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607
[9] Aghamohammadi A, Khorrami M and Shariati A 1995 *J. Phys. A: Math. Gen.* **28** L225
[10] Manin Yu I 1988 *Preprint* CRM-1561
[11] Corrigan E, Fairlie B, Fletcher P and Sasaki R 1990 *J. Math. Phys.* **31** 776
[12] Celik S 1996 *J. Math. Phys.* **37** 3568
[13] Celik S 1996 Hopf algebra structure of $Gr_q(1|1)$ and h-deformation of $Gr(1|1)$ *Preprint* MSUMB 96-02